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Analytic solutions for potential flow over a class of semi-infinite two-dimensional bodies having circular-arc noses

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A new class of analytic solutions to the problem of two-dimensional potential flow is presented here. The method of solution has features of both direct and indirect solutions. The bodies about which flow is computed are semi-infinite and have forward regions that either are flat or consist of a circular arc, which may be convex or concave to the flow. Closed-form solutions are obtained for the surface velocity. Afterbody shapes are defined by implicit equations containing a quadrature. Certain analytic properties of the solutions are investigated. An interesting feature of the bodies is the presence of a 'pseudo corner' where the slope angle is continuous but the curvature is infinite. The surface velocity becomes logarithmically infinite at these points in contrast to the power-law behaviour at a true corner. One case of the convex circular arc has finite velocity everywhere, and in some sense represents flow over a circular cylinder with a 'natural' separation point. This point occurs at 77·45° from the front stagnation point, which is close to the separation point for incompressible laminar boundary-layer flow.

1. Introduction

The problem of incompressible potential flow about a solid body has long been of interest in fluid mechanics because of the surprisingly close agreement between this simple flow and real low-speed flow without boundary-layer separation. For completely general bodies even the problem of potential flow is not easy, and resort must be made to a large-scale computer and to numerical techniques (Hess & Smith 1966; Hess 1971). For a few bodies the potential-flow velocity field may be expressed in terms of a closed analytic expression or at worst an expression, such as an infinite series or a quadrature, whose evaluation is simpler than a complete numerical solution. Such solutions have always been of interest as illustrative examples of potential flow. However, they are also of practical interest in at least two ways. First, quick approximate studies may be conducted using these analytic expressions to establish the feasibility of a proposed design procedure. For example, the flow field of an elliptic cylinder may be used to approximate that of a two-dimensional airfoil of the same thickness for the purpose of estimating far-field interference effects. Second, the exact analytic solution for a particular body may be used to evaluate the accuracy of general numerical methods, such as those reviewed in Hess (1971). Numerical methods may be refined to yield higher accuracy with an increase in computing time. Comparison with an exact solution indicates how far the refinement process need be carried to obtain acceptable accuracy without unnecessarily increasing computing time. Of course the situation is affected by the geometry of the body in question, so it is useful to have analytic solutions for as many types of bodies as possible.

Analytic solutions have been obtained in two ways: direct solution and indirect solution. In the former the body is specified to begin with, and the flow field must be calculated. This is of course the problem solved by general numerical methods for arbitrary bodies. However, for a very small number of bodies such solutions may be obtained by analytic techniques, principally separation of variables or conformal mapping, the latter of which is restricted to two-dimensional problems. A larger class of solutions can be generated by indirect solutions. These are based on the use of flow singularities, such as sources, vortices and dipoles. Each singularity gives rise to a velocity field that satisfies the basic potential-flow equations except at the singularity itself. Such flows are superimposed upon a uniform stream. Any streamline of the resulting flow may be considered as the boundary of a body, the flow about which is given by adding the individual flows of the singularities and the uniform stream. Proper distribution of singularities and proper selection of a streamline yield flows about interesting families of bodies. In contrast to the direct problem, for which the body is given, calculation of the body shape is the most difficult part of an indirect solution. The class of analytic solutions to be described here is generated by a method that has features of both the direct and the indirect solutions.

The idea of obtaining indirect solutions by superposition of point sources was put forward by Rankine (1871). Because this method has several features in common with the method to be described here, it is useful to outline the simplest case of Rankine's procedure, a single two-dimensional point source (three-dimensional line source) in a uniform stream. The velocity at a point (x, y) due to a source of unit strength at a point (ξ, η) is

$$\mathbf{V}_e = (2/r^2)\,[(x-\xi)\,\mathbf{i}+(y-\eta)\,\mathbf{j}], \tag{1}$$

where

$$r^2 = (x-\xi)^2+(y-\eta)^2 \tag{2}$$

and $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along the x and y axes, respectively. With this definition of the unit source the flux of velocity from a source per unit time is $4\pi K$, where K is the strength of the source. The simplest Rankine-type flow is obtained from a uniform stream of velocity U parallel to the x axis and a point source of strength K at the origin. The velocity field is

$$\mathbf{V} = \left(U+\frac{2Kx}{x^2+y^2}\right)\mathbf{i}+\frac{2Ky}{x^2+y^2}\,\mathbf{j}. \tag{3}$$

The velocity is zero at the point

$$x_0 = -2K/U, \quad y_0 = 0. \tag{4}$$

The streamline which bifurcates at this stagnation point is taken as a body contour and is sketched in figure 1. The body is semi-infinite and symmetric

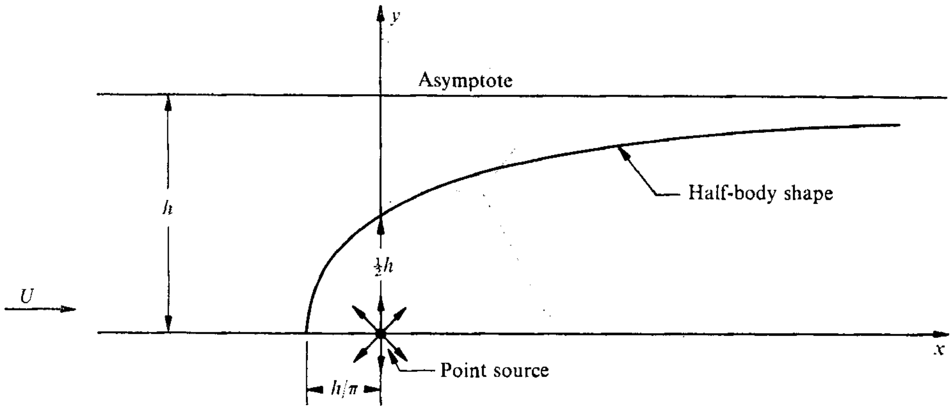

FIGURE 1. Two-dimensional body shape obtained by superposing a point source and a uniform stream.

about the x axis. All fluid emitted by the source remains within the body, so for large positive values of x, for which the velocity is U, the half-thickness h of the body is given by

$$h = 2\pi K/U. \tag{5}$$

It can be shown (Milne-Thomson 1950, p. 196) that the shape of the body is given by the equation

$$\frac{1}{x} = \frac{1}{y}\tan\left(\frac{\pi y}{h}\right), \tag{6}$$

which is the shape plotted in figure 1. Thus in evaluating a numerical method the shape (6) would be input to the method and the calculated velocity compared with (3). However, for interference studies, often the details of the shape are not required. It is enough to know that a semi-infinite body of thickness (5) with its nose located at (4) gives rise to the velocity field (3). Calculation of (6) is not necessary.

2. A new type of solution

Hess & Smith (1966) describe a method for solving the problem of two-dimensional potential flow directly by means of a source distribution over the body in question. Specifically, a source distribution of variable strength σ lies on the body. The velocity at any point is

$$\mathbf{V} = \oint \mathbf{V}_e \sigma\, ds + \mathbf{U}, \tag{7}$$

where s is arc length along the body, $\mathbf{U}$ is the free-stream velocity and $\mathbf{V}_e$ is given by (1) with (ξ, η) identified with a general point of the body surface. The integral in (7) is over the entire body, which is either a closed curve or a semi-infinite shape of the type illustrated in figure 1. Both these cases are called 'closed'

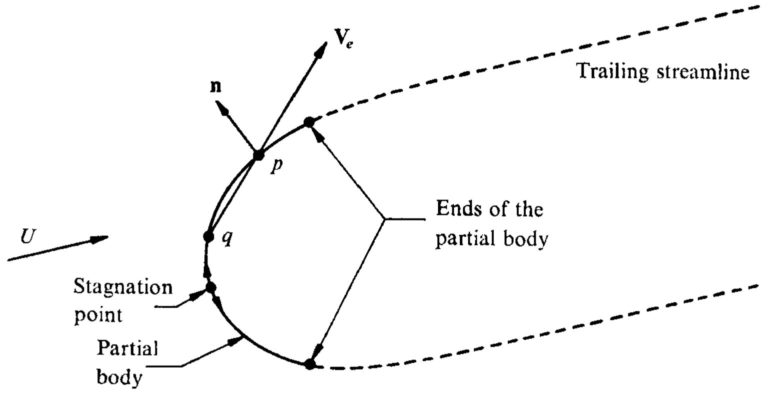

FIGURE 2. Generation of solutions using a source density on a partial body.

bodies, and the notation for the integral in (7) reflects this. The condition of zero normal velocity on the boundary is

$$2\pi\sigma(p) + \oint (\mathbf{V}_e . \mathbf{n})\, \sigma(q)\, ds = -\mathbf{n} . \mathbf{U}, \tag{8}$$

where p denotes the point (x, y), which is a point on the body, and q denotes the point (ξ, η), as illustrated in figure 2. The unit outward normal vector at p is $\mathbf{n}$. The origin of the term $2\pi\sigma(p)$ in (8) is discussed by Hess & Smith (1966). Equation (8) is an integral equation for σ. Once it has been solved velocities are computed from (7). In general the integrand of (8) depends on the co-ordinates of both p and q, and a numerical solution is the only possibility. An analytic solution is possible for a circle, but of course this solution can be obtained by other means.

In general terms the present method of generating analytic solutions parallels the above direct solution with just one difference. While the general method above distributes source density on a *complete closed* body, the new method distributes sources on a *partial open* body as shown in figure 2. The condition of zero normal velocity is applied over the partial body to obtain the integral equation

$$2\pi\sigma(p) + \oint (\mathbf{V}_e . \mathbf{n})\, \sigma(q)\, ds = -\mathbf{n} . \mathbf{U}, \tag{9}$$

where the integral notation reflects the fact that the integration is over a non-closed curve. When σ is known the velocity field is calculated from

$$\mathbf{V} = \oint \mathbf{V}_e \sigma\, ds + \mathbf{U}. \tag{10}$$

The only difference between the set of equations (9) and (10) and the set (7) and (8) is the domain of integration. Solution of (9) presents no difficulties in principle or numerically.

A question arises as to the physical significance of the solution represented by (9) and (10). It is clear that the partial body of figure 2 is a streamline of the flow, which in the general case shown has a forward stagnation point where the streamline bifurcates. If the source density is non-negative at the end of the body the streamline simply leaves the partial body and continues downstream. This portion is called the trailing streamline. In general there is a net positive source strength, so the streamlines from the two ends of the body do not rejoin each other but proceed to infinity a finite distance apart. Thus the velocity field (10) may be thought of as that due to a semi-infinite body consisting of the partial body and the two streamlines leaving it (figure 2). The source distribution on the complete body is that from (9) on the partial body and zero on the streamlines.

If the source density is zero at an end of the partial body the velocity is finite there. If the source density is positive there, the velocity is infinite in a manner qualitatively similar to that in flow about a sharp convex corner. However, the singularity is logarithmic here, rather than being the power-law singularity appropriate to a corner. In this case the streamline leaving the body does so with continuous slope but with infinite curvature. If the source density is negative at an end of the partial body, the velocity there is infinite and in the upstream direction. Thus the streamline does not leave the body but joins it from downstream, and there must be additional stagnation points in the neighbourhood. Apparently, a case having negative source density at an end of the partial body does not have the same physical significance as a case where the source density there is positive.

For the same body shape, solution of integral equation (9) is no easier than solution of integral equation (8). Analytic solutions can be obtained only for a circular arc and for a straight line. However, these represent a one-parameter family of bodies, instead of the single body for which (8) is analytically solvable. Furthermore, these are new analytic solutions.

3. Specific analytic solutions

3.1. *Flat-nosed body*

Consider a partial body consisting of a straight line lying along the y axis from $-d$ to $+d$, as illustrated in figure 3 (*a*). The exterior of the body is $x = 0-$, and the unit normal vector is $-\mathbf{i}$. From (1) with $x = \xi = 0$ it follows that

$$\mathbf{V}_e . \mathbf{n} = 0, \tag{11}$$

so that the integral term of (9) vanishes, and the equation can be solved trivially. If the free stream makes an angle α with the x axis, then

$$\mathbf{U} = U[\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j}] \tag{12}$$

and the solution of (9) is

$$\sigma = (2\pi)^{-1}\cos\alpha. \tag{13}$$

Integration of this over the straight line can be accomplished by the methods of Hess & Smith (1966) to give the velocity field

$$\left.\begin{aligned} V_x &= U\cos\alpha\left\{\frac{1}{\pi}\left[\tan^{-1}\left(\frac{d-y}{x}\right)+\tan^{-1}\left(\frac{d+y}{x}\right)\right]+1\right\}, \\ V_y &= U\left[\frac{\cos\alpha}{2\pi}\log\left(\frac{(d+y)^2+x^2}{(d-y)^2+x^2}\right)+\sin\alpha\right]. \end{aligned}\right\} \tag{14}$$

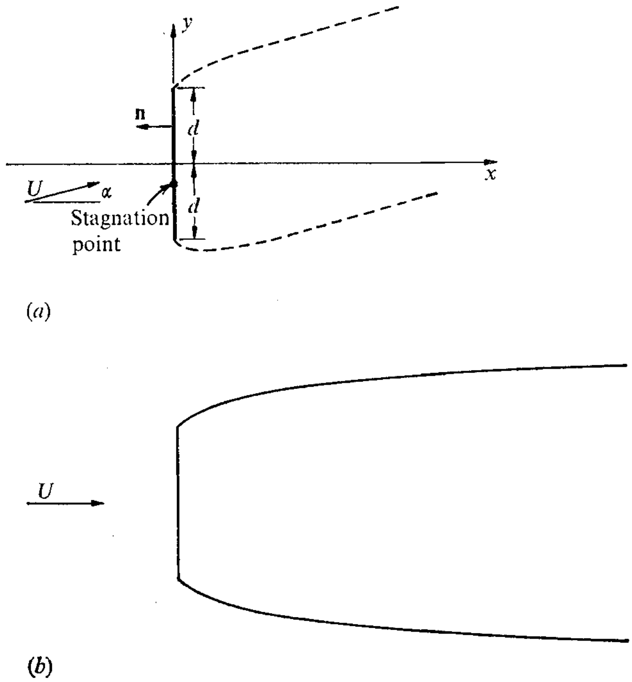

FIGURE 3. A body with a flat nose. (a) General notation. (b) Calculated body shape for the symmetric case ($\alpha = 0$).

On the front of the body ($x = 0-$), $V_x = 0$ as required, and

$$V_y = U\left[\frac{\cos\alpha}{\pi}\log\left(\frac{1+y/d}{1-y/d}\right)+\sin\alpha\right]. \quad (15)$$

The stagnation point is at

$$y_0/d = -\tanh\left(\tfrac{1}{2}\pi\tan\alpha\right). \quad (16)$$

The most interesting case is the symmetric one, for which $\alpha = 0$. The implicit equation of the trailing streamline on the upper side of the body is

$$\tfrac{3}{2}\pi P - P\tan^{-1}P + \tfrac{1}{2}\log(1+P^2) = \tfrac{1}{2}\pi Q - Q\tan^{-1}Q + \tfrac{1}{2}\log(1+Q^2), \quad (17)$$

where

$$P = (y-d)/x, \quad Q = (y+d)/x. \quad (18)$$

This curve is sketched in figure 3(b). Tabulated values of the streamline shape and its velocity are given in table 1. The final width of the body is just twice its initial width, i.e. as $x \to \infty$, $y \to 2d$.

In the neighbourhood of the point $x = 0$, $y = d$, the trailing streamline has the behaviour

$$y - d \sim -\frac{x}{\pi}\log\frac{x}{2d}, \quad \frac{dy}{dx} \sim -\frac{1}{\pi}\log\frac{x}{2d}, \quad \frac{d^2y}{dx^2} \sim -\frac{1}{\pi}\frac{1}{x}. \quad (19)$$

Thus the slope angle of the streamline approaches $\frac{1}{2}\pi$ and there is no discontinuity in this quantity. However, the curvature is infinite. The logarithmic infinity in surface velocity (15) is apparently the appropriate singularity for this 'pseudo corner', i.e. a weaker singularity than for any true corner.

x/d	y/d	V_T/U	x/d	y/d	V_T/U
0	1·0	∞	4·5025	1·7600	1·1223
0·0501	1·0555	1·4403	4·7506	1·7696	1·1173
0·0997	1·0937	1·4070	5·0100	1·7806	1·1125
0·1497	1·1262	1·3793	5·2507	1·7881	1·1083
0·2507	1·1810	1·3489	5·5021	1·8006	1·1042
0·3506	1·2261	1·3307	5·7504	1·8062	1·1004
0·4507	1·2648	1·3171	6·0024	1·8151	1·0968
0·5511	1·3009	1·3051	6·5820	1·8284	1·0895
0·6508	1·3300	1·2959	7·0077	1·8312	1·0848
0·8000	1·3760	1·2811	7·4906	1·8539	1·0798
0·9497	1·4122	1·2692	7·9936	1·8617	1·0753
1·0989	1·4406	1·2589	8·4926	1·8705	1·0713
1·2500	1·4750	1·2476	9·0050	1·8726	1·0676
1·4519	1·5045	1·2354	9·5012	1·8741	1·0644
1·6502	1·5413	1·2229	10·0000	1·8800	1·0614
1·8484	1·5693	1·2120	11·0000	1·8843	1·0561
2·0513	1·5949	1·2017	12·0000	1·8939	1·0517
2·2523	1·6126	1·1925	13·0000	1·9021	1·0479
2.5000	1.6375	1·1818	14·0000	1·9091	1·0446
2·7510	1·6548	1·1721	15·0000	1·9151	1·0417
2·9985	1·6837	1·1627	16·5000	1·9228	1·0380
3·2499	1·6974	1·1546	18·5000	1·9312	1·0340
3·5026	1·7145	1·1469	21·0000	1·9394	1·0300
3·7523	1·7317	1·1399	24·0000	1·9469	1·0263
3·9944	1·7442	1·1337	29·0000	1·9561	1·0219
4·2517	1·7509	1·1278	∞	2·0000	1·0000

TABLE 1. Trailing streamline for the flat-nosed body

In terms of the length s along the body measured from the singularity the curvature κ behaves locally as

$$\kappa \sim [-1/\pi s(\log s)^2]. \tag{20}$$

For classical free-streamline shapes (Milne-Thomson 1950, p. 293) the curvature of the streamline near the point where it leaves the body has the local behaviour

$$\kappa \sim \text{constant}/\surd s. \tag{21}$$

The singularity of (21) is less severe than that of (20), and the velocity on a free-streamline shape is not infinite but has an infinite derivative at the point of streamline departure.

3.2. *Concave circular arc*

Integral equation (9) can be solved if the partial body is a circular arc. This section considers the case where the arc is concave to the flow. The arc is assumed to have radius a with centre at the origin and to be symmetric about the x axis with angular extent from $-\beta$ to β (figure 4). The velocity at a point on the circle at an angle θ due to a unit point source at an angle ϕ (all angles measured from the positive x axis) is from (1)

$$\mathbf{V}_e = (2a/r^2)\,[(\cos\theta - \cos\phi)\,\mathbf{i} + (\sin\theta - \sin\phi)\,\mathbf{j}], \tag{22}$$

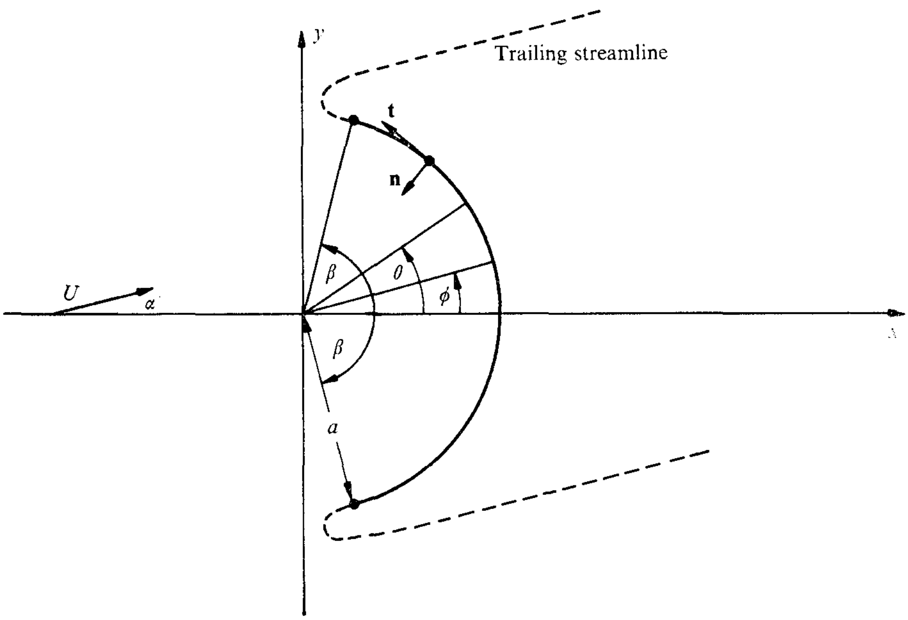

FIGURE 4. Notation for a concave circular arc.

where r^2 from (2) is given by

$$r^2 = 2a^2[1-\cos(\theta-\phi)]. \tag{23}$$

The outward normal vector for the concave arc is

$$\mathbf{n} = -\cos\theta\mathbf{i}-\sin\theta\mathbf{j}. \tag{24}$$

Multiplying (22) and (24) gives

$$\mathbf{V}_e.\mathbf{n} = -1/a. \tag{25}$$

The kernel of integral equation (8) or (9) is constant for a circle and negative for the concave case. As before the free stream is assumed to make an angle α with the x axis, equation (12). Now with $ds = a\,d\phi$ the integral equation for the source distribution is

$$2\pi\sigma(\theta)-\int_{-\beta}^{\beta}\sigma(\phi)\,d\phi = U\cos(\theta-\alpha). \tag{26}$$

Since the integral is independent of θ the solution of (26) is clearly of the form

$$\sigma(\theta) = (U/2\pi)\cos(\theta-\alpha)+\text{constant}, \tag{27}$$

which when put into (26) gives

$$\sigma(\theta) = \frac{U}{2\pi}\left[\cos(\theta-\alpha)+\frac{\sin\beta\cos\alpha}{\pi-\beta}\right]. \tag{28}$$

The tangential velocity V_T on the circular arc is obtained from the dot product of (22) with the tangential vector

$$\mathbf{t} = -\sin\theta\mathbf{i}+\cos\theta\mathbf{j} \tag{29}$$

in the form
$$V_T = \int_{-\beta}^{\beta} \frac{\sin(\theta-\phi)}{1-\cos(\theta-\phi)}\,\sigma(\phi)\,d\phi - U\sin(\theta-\alpha), \tag{30}$$

which after some manipulation gives

$$V_T/U = \frac{1}{\pi}\left[\cos(\theta-\alpha)+\frac{\sin\beta\cos\alpha}{\pi-\beta}\right]\log\left(\frac{\tan\frac{1}{2}\beta+\tan\frac{1}{2}\theta}{\tan\frac{1}{2}\beta-\tan\frac{1}{2}\theta}\right) + \left(\frac{\beta}{\pi}-1\right)\sin(\theta-\alpha)-\pi^{-1}\sin\beta\sin\alpha. \tag{31}$$

For non-negative α the source density is always positive at $\theta = \beta$, and the tangential velocity is $+\infty$ there. For small enough α, the same is true for $\theta = -\beta$. However for $\alpha = \alpha_m$ defined by

$$\tan\alpha_m = \cot\beta + [1/(\pi-\beta)] \tag{32}$$

the source density is zero and V_T is finite at $\theta = -\beta$. For $\alpha > \alpha_m$ the source density is negative at $\theta = -\beta$, V_T is $-\infty$, and the flow is not physically meaningful. However, the flow is always meaningful for the symmetric case $\alpha = 0$.

This type of analytic solution is direct in its treatment of the circular arc but indirect in its treatment of the trailing streamline. As in any indirect method, the hardest part of the calculation is finding the shape of the trailing streamline. This situation is evident in the case of the flat-nosed body of §3.1. For the circular-arc bodies calculation of the trailing streamline is still more difficult, but it can be done by requiring equal velocity fluxes at various x locations: the resulting equation for y as a function of x along the streamline is implicit and contains a quadrature. Calculation of the trailing streamline for the symmetric case $\alpha = 0$ is discussed in §4, and some shapes are shown in figure 8. One quantity that is easy to compute is the final width of the body, which is

$$W = \frac{4\pi a}{\pi-\beta}\sin\beta\cos\alpha. \tag{33}$$

For the frequently useful case of symmetric ($\alpha = 0$) flow about a concave semi-circle ($\beta = \frac{1}{2}\pi$), the final width is four times the diameter of the semi-circle.

Given the source density (28) the velocity off the surface at points of the field can be calculated, but this will not be pursued here. However, it is interesting to compute the velocity at the centre of the circular arc to show how small velocities in a concave region usually are. For $\alpha = 0$ the velocity at the origin is in the x direction and has magnitude given by

$$V(0,0)/U = 1-\frac{1}{2\pi}\left[2\beta+\sin 2\beta+\frac{4}{\pi+\beta}\sin^2\beta\right]. \tag{34}$$

For the semi-circle, $\beta = \frac{1}{2}\pi$, this is given by

$$V(0,0)/U = \tfrac{1}{2}-(4/\pi^2) \approx 0{\cdot}095, \tag{35}$$

less than 10 % of the free-stream velocity.

3.3. *Convex circular arc*

Again the arc is symmetric about the x axis, has radius a, and has an angular extent from $-\beta$ to $+\beta$. However, as shown in figure 5, it is convenient to measure

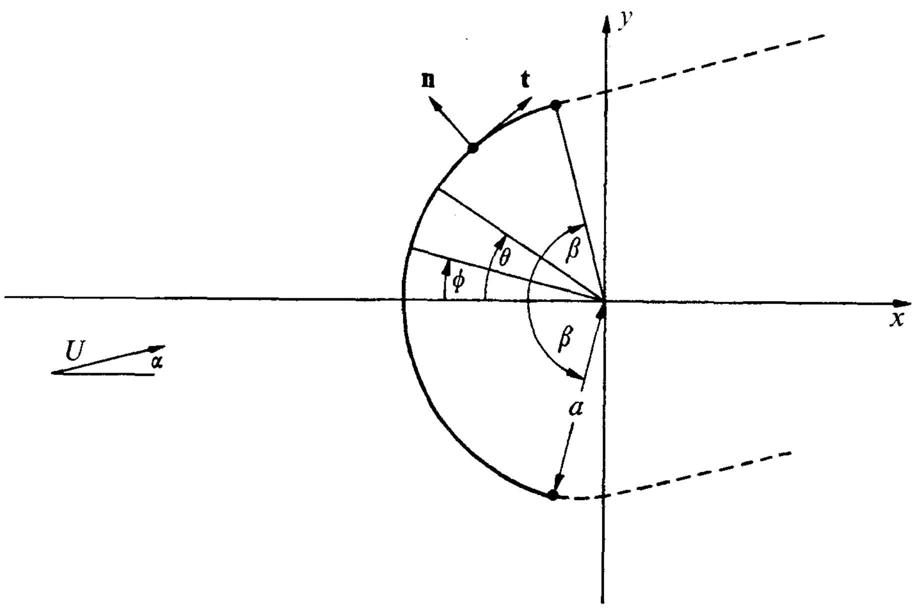

FIGURE 5. Notation for a convex circular arc.

the angles θ and ϕ from the negative-x axis. With this notation $\mathbf{V}_e$ is given by (22) with the sign of the x component changed, and the unit outward normal vector is

$$\mathbf{n} = -\cos\theta\mathbf{i} + \sin\theta\mathbf{j}. \tag{36}$$

The kernel of integral equation (9) is

$$\mathbf{V}_e . \mathbf{n} = 1/a, \tag{37}$$

so that the integral equation is

$$2\pi\sigma(\theta) + \int_{-\beta}^{\beta} \sigma(\phi)\, d\phi = U\cos(\theta+\alpha). \tag{38}$$

The solution for the source density is

$$\sigma(\theta) = \frac{U}{2\pi}\left[\cos(\theta+\alpha) - \frac{\sin\beta\cos\alpha}{\pi+\beta}\right]. \tag{39}$$

The tangential velocity on the circular arc is given by

$$V_T/U = \int_{-\beta}^{\beta} \frac{\sin(\theta-\phi)}{1-\cos(\theta-\phi)}\, \sigma(\phi)\, d\phi + \sin(\theta+\alpha), \tag{40}$$

which becomes

$$V_T/U = \frac{1}{\pi}\left[\cos(\theta+\alpha) - \frac{\sin\beta\cos\alpha}{\pi+\beta}\right] \log\left(\frac{\tan\frac{1}{2}\beta + \tan\frac{1}{2}\theta}{\tan\frac{1}{2}\beta - \tan\frac{1}{2}\theta}\right)$$
$$+ \left(\frac{\beta}{\pi} + 1\right)\sin(\theta+\alpha) + \pi^{-1}\sin\beta\sin\alpha. \tag{41}$$

As for the concave cases the details of the streamline shape are discussed in §4 for the symmetric case $\alpha = 0$. The final width of the body is

$$W = \frac{4\pi a}{\pi+\beta}\sin\beta\cos\alpha. \tag{42}$$

Just as for the concave case there is a value of α above which the source density at $\theta = \beta$ is negative and the flow is not physically meaningful. However, the situation is more complicated because sufficiently large values of β make the source density negative at $\theta = \beta$ even for $\alpha = 0$.

Principal attention has been directed to the symmetric case $\alpha = 0$. For such flows the source density, which is also the coefficient of the logarithm in the expression for the tangential velocity, is

$$\sigma(\theta) = \frac{U}{2\pi}\left[\cos\theta - \frac{\sin\beta}{\pi+\beta}\right]. \tag{43}$$

In particular at $\theta = \beta$

$$\sigma(\beta) = \frac{U}{2\pi}\left[\cos\beta - \frac{\sin\beta}{\pi+\beta}\right]. \tag{44}$$

It is clear that $\sigma(\beta)$ is positive for small values of β but is negative for $\beta = \frac{1}{2}\pi$. Thus there is some value $\beta_m < \frac{1}{2}\pi$ such that for $\beta > \beta_m$ the flows are not physically significant. For $\beta < \beta_m$ the situation is qualitatively similar to that described in §3.1 for the flat-nosed body. Namely, the surface velocity is logarithmically infinite at the end of the circular arc and the streamline curvature is infinite there. The rather interesting case $\beta = \beta_m$ is described in the next section.

3.4. *A special case*

The value $\beta = \beta_m$, which makes $\sigma(\beta) = 0$ for the symmetric case $\alpha = 0$, is the largest value of β that gives meaningful flow. It is defined as the solution of the equation

$$\tan\beta_m = \pi+\beta_m. \tag{45}$$

Solution of this equation gives approximately

$$\beta_m = 77{\cdot}45^\circ. \tag{46}$$

The convex circular arc of extent $2\beta_m$ at $\alpha = 0$ is the only body of the class discussed here for which the velocity is finite everywhere. In this sense $\theta = \beta_m$ is a 'natural' point of separation of incompressible flow from a circular cylinder. It is interesting to note how close this is to the experimentally determined point of laminar separation, which various experiments have given as between 75° and 90°. It is clear that the value of β which makes (44) zero maximizes the width (42). Thus the body with $\beta = \beta_m$ has the maximum asymptotic width

$$W = 4\pi a\cos\beta_m, \tag{47}$$

which is just 2π times the width of the body at the point where the streamline leaves the circular arc. This body is sketched in figure 7 (*a*) below along with other members of the family. Co-ordinates of the trailing streamline and its surface

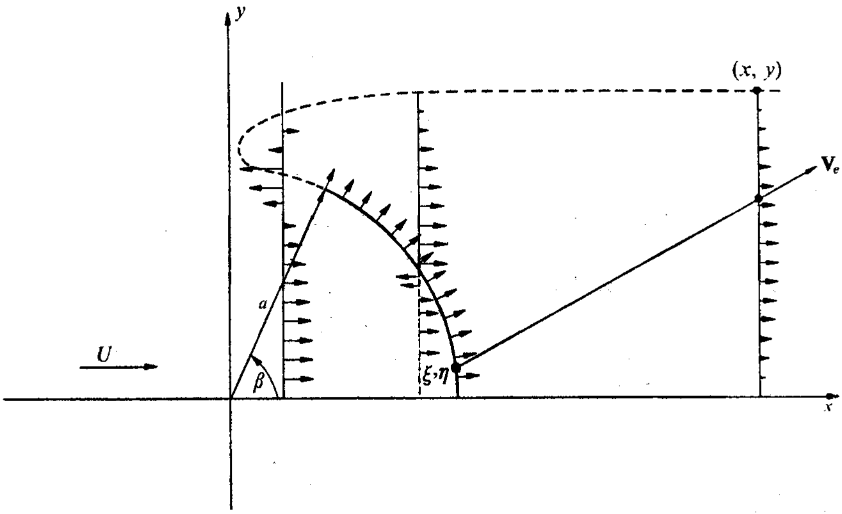

FIGURE 6. Control volumes used to compute the trailing streamline.

velocity distribution are given in table 2. The velocity distribution on the circular arc, which is obtained from (41), attains its maximum value of 1·4418 at an angular location of $\theta = 73{\cdot}59°$ and falls to a value of 1·3961 at $\theta = \beta_m$.

4. Calculation of the trailing streamline for the symmetric case

The location of the trailing streamline is determined by equating to zero the total flux of velocity out of a control volume. The basic quantity involved in this calculation is the flux through a vertical line from 0 to y at a location x due to a unit point source at (ξ, η) (figure 6). This flux is

$$S(x,y,\xi,\eta) = \int_0^y (\mathbf{V}_e.\mathbf{i})\,dy = 2\left[\tan^{-1}\left(\frac{y-\eta}{x-\xi}\right)+\tan^{-1}\left(\frac{\eta}{x-\xi}\right)\right], \qquad (48)$$

where the inverse tangents are to be evaluated in the range $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$. Thus the velocity flux due to the entire circular arc through this vertical line is

$$F(x,y;\beta) = a\int_{-\beta}^{\beta} S(x,y,\xi,\eta)\,\sigma(\theta)\,d\theta. \qquad (49)$$

The flux of fluid out of the downstream side of the circular arc above the x axis is

$$N(\beta) = 4\pi a\int_0^{\beta}\sigma(\theta)\,d\theta = \tfrac{1}{2}UW(\beta), \qquad (50)$$

where W is the final width of the body. For points downstream of the circular arc the control volume is bounded by (i) the streamline itself, (ii) the x axis (across which there is no flow by symmetry), (iii) the downstream side of the circular arc, and (iv) the vertical line at x from 0 to y on the streamline (figure 6). The equation relating x and y is

$$F(x,y;\beta)+Uy-\tfrac{1}{2}UW(\beta) = 0, \qquad (51)$$

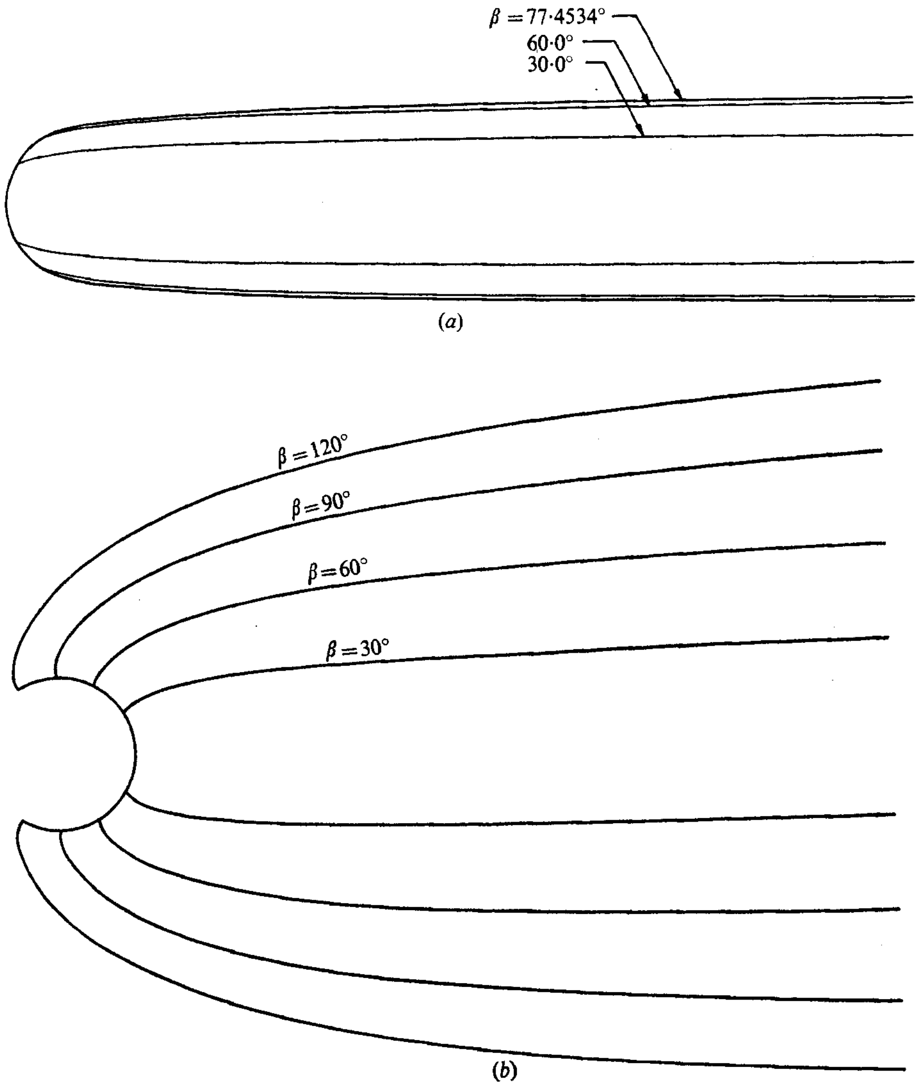

FIGURE 7. Semi-infinite bodies having (*a*) convex and (*b*) concave circular-arc noses for the symmetric case ($\alpha = 0$).

where Uy is the free-stream flux across the vertical line. Equation (50) is valid for both convex and concave circular arcs. The quantities ξ and η as functions of θ, W as a function of β, and σ as a function of θ and β are those appropriate to the convex or concave case, respectively. For the convex case (51) represents the entire trailing streamline from the point where it leaves the circle to infinity. For the concave arc (51) is only valid downstream of the arc, i.e. for $x > a$ (figure 6). There are two other regimes, where different expressions are required.

For points above the circular arc, i.e. for $\cos\beta < x/a < 1$, the control volume is bounded by (i) the streamline itself, (ii) the portion of the downstream side of the circular arc upstream of the vertical line at x and (iii) the vertical line at x

x/a	y/a	V_T/U	x/a	y/a	V_T/U
−0·2172	0·9761	1·3961	1·7497	1·1839	1·1391
−0·1972	0·9805	1·3732	1·9664	1·1944	1·1310
−0·1752	0·9852	1·3574	2·2048	1·2048	1·1231
−0·1510	0·9902	1·3437	2·4670	1·2148	1·1154
−0·1244	0·9956	1·3314	2·7554	1·2246	1·1078
−0·0951	1·0013	1·3199	3·0726	1·2340	1·1005
−0·0629	1·0073	1·3090	3·4216	1·2430	1·0935
−0·0275	1·0137	1·2986	3·8055	1·2517	1·0868
0·0115	1·0205	1·2885	4·2278	1·2599	1·0804
0·0544	1·0277	1·2788	4·6923	1·2677	1·0743
0·1015	1·0353	1·2693	5·2032	1·2751	1·0686
0·1534	1·0433	1·2600	5·7653	1·2820	1·0632
0·2105	1·0516	1·2508	6·3836	1·2885	1·0581
0·2732	1·0603	1·2419	7·0636	1·2946	1·0534
0·3423	1·0694	1·2330	7·8117	1·3003	1·0490
0·4182	1·0788	1·2242	8·6346	1·3055	1·0449
0·5018	1·0885	1·2155	9·5398	1·3105	1·0411
0·5937	1·0985	1·2068	10·5355	1·3150	1·0377
0·6947	1·1088	1·1982	11·6308	1·3192	1·0344
0·8059	1·1193	1·1896	12·8356	1·3231	1·0315
0·9283	1·1299	1·1811	14·1609	1·3266	1·0287
1·0628	1·1407	1·1726	15·6187	1·3299	1·0262
1·2108	1·1515	1·1641	17·2223	1·3329	1·0239
1·3736	1·1624	1·1557	18·9862	1·3357	1·0218
1·5527	1·1732	1·1474	20·9266	1·3382	1·0199
			∞	1·3647	1·0000

TABLE 2. Trailing streamline for the body whose nose is a convex circular arc of semi-angle $\beta = 77{\cdot}45°$

between the arc and the streamline. The equation relating x and y on the streamline is

$$F(x,y;\beta)+U\left[y+\sin\psi+2\frac{\sin\psi}{\pi-\beta}(\psi-\pi)\right] = 0, \tag{52}$$

where
$$\psi = \cos^{-1}(x/a). \tag{53}$$

The flux through part (iii) above, the vertical line at x between the circular arc and the streamline, equals $F(x,y;\beta)-F(x,a\sin\psi;\beta)$. However, $F(x,a\sin\psi;\beta)$, which represents flux through the dotted vertical line of figure 6, is zero because the circular arc is a streamline and is thus omitted from (52). In this regime $\xi-x$ may have either sign, and the integrand for $F(x,y;\beta)$ is discontinuous for $\theta = \pm\psi$. For good accuracy in evaluating the integral the regions between the discontinuities should be evaluated separately.

Finally for points on the streamline upstream of the circular arc, i.e. for $x/a < \cos\beta$, the control volume is bounded by (i) the streamline itself, (ii) the upstream side of the circular arc, (iii) the x axis and (iv) the vertical line at x from 0 to y on the streamline. Since the first three are streamlines, the condition is that the flux across the vertical line vanish, namely

$$F(x,y;\beta)+Uy = 0. \tag{54}$$

x/a	y/a	V_T/U	x/a	y/a	V_T/U
0	1·0	∞	0·7000	2·0076	1·2287
−0·0092	1·0107	0·9361	0·8000	2·0723	1·2400
−0·0155	1·0241	0·8862	0·9000	2·1324	1·2489
−0·0200	1·0381	0·8695	1·0	2·1886	1·2559
−0·0230	1·0530	0 8654	1·1050	2·2438	1·2617
−0·0252	1·0711	0·8668	1·2321	2·3061	1·2667
−0·0259	1·0931	0·8740	1·3858	2·3758	1·2707
−0·0252	1·1130	0·8825	1·5718	2·4528	1·2731
−0·0230	1·1345	0·8933	1·7969	2·5370	1·2732
−0·0200	1·1547	0·9038	2·0692	2·6279	1·2707
−0·0155	1·1767	0·9159	2·3988	2·7245	1·2651
−0·0092	1·2021	0·9297	2·7975	2·8256	1·2561
0·0	1·2322	0·9463	3·2800	2·9296	1·2438
0·010	1·2602	0·9614	3·8638	3·3035	1·2116
0·030	1·3078	0·9865	4·5701	3·3138	1·2019
0·050	1·3486	1·0073	5·4249	3·2391	1·1903
0·075	1·3939	1·0292	6·4591	3·3346	1·1695
0·100	1·4345	1·0481	7·7105	3·4234	1·1488
0·150	1·5067	1·0795	9·2247	3·5045	1·1289
0·200	1·5703	1·1050	11·0569	3·5772	1·1104
0·250	1·6279	1·1264	13·2738	3·6414	1·0937
0·300	1·6808	1·1446	15·9563	3·6974	1·0790
0·400	1·7762	1·1741	19·2022	3·7458	1·0662
0·500	1·8609	1·1968	21·0724	3·7672	1·0605
0·600	1·9375	1·2146	∞	4·0	1·0000

TABLE 3. Trailing streamline for the body whose nose is a concave circular arc of semi-angle $\beta = 90°$

For each x, equation (54) yields two values of y, in accordance with the fact that the streamline is double valued in the region, until a certain minimum x is reached below which no value of y makes the left side of (54) zero (figure 6).

Equations (48), (50) and (51) may appear formidable, but they can be solved quite easily on a desk calculator if only a moderate number of points on the streamline are required. Some representative shapes for the convex case are shown in figure 7 (*a*), and shapes for the concave case are shown in figure 7 (*b*). Streamline shapes for the convex circular arc with $\beta = \beta_m = 77{\cdot}45°$ and for the concave semi-circle are tabulated in tables 2 and 3, respectively. Also tabulated are the surface velocity distributions on these streamlines.

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